Algebra II — Lesson 3 - Long Division

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1. Integer Long Divison

It is important to realize that there are two forms of long division; integer division and decimal division.

With integer division, we take two integers, divide one into the other, and obtain two integers. It is oddly enlightening to review this process.

We will divide 6 into 5217. We ask "does six go into five". The answer is no. Then ask "does six go into 52"? Yes, it does; it goes 8 times. Write the 8 upstairs. Multiply $6 \times 8 = 48$ and subtract $52 - 48 = 4$. Write the 4 at the bottom.

This 4 lines up under the 2 in 52. We pull the 1 from 5217 down, and now have 41. We ask if 6 goes into 41. It goes 6 times. Multiple $6 \times 6 = 36$ and subtract $41 - 36 = 5$. Repeat this process until we use up all the digits.

$$6\overline{)5217} \qquad \begin{array}{r} 8 \\ 6\overline{)5217} \\ \underline{48} \\ 4 \end{array} \qquad \begin{array}{r} 86 \\ 6\overline{)5217} \\ \underline{48} \\ 41 \\ \underline{36} \\ 5 \end{array} \qquad \begin{array}{r} 869 \\ 6\overline{)5217} \\ \underline{48} \\ 41 \\ \underline{36} \\ 57 \\ \underline{54} \\ 3 \end{array}$$

Notice that after each subtraction, the result is less than 6. This is because if we had ended up with a difference greater than six, we would have been able to use a larger multiple in the quotient.

Now, let us see how this looks if we pull down all the digits as we go.

$$\begin{array}{r} 869 \\ 6\overline{)5217} \\ \underline{4800} \\ 417 \\ \underline{360} \\ 57 \\ \underline{54} \\ 3 \end{array}$$

Each stage of long division allows us to rewrite 5217 as a product of six plus some remainder. After the first stage, we have computed that $5217 = 6 \cdot 800 + 417$. After the second, we have $5217 = 6cross860 + 57$. Finally, the last result gives us $5217 = 6 * 869 + 3$. We stop when the remainder is less than the number we divided by.

The underlying theorem we need to understand this is commonly known as the *Division Algorithm*.

Proposition 1. (Division Algorithm for Integers)
Let m and n be integers. Then there exist unique integers q and r such that

$$n = mq + r \quad \text{and } 0 \leq r < m.$$

This equation applies when we divide m into n. We call n the *dividend*, m is the *divisor*, q is the *quotient*, and r is the *remainder*. The output of integer division consists of two integers; the quotient and the remainder.

We know this theorem is true, because we know how to obtain q and r using long division.

2. Decimal Long Division

If we wish to find the decimal expansion of a fraction, we use decimal long division. Here, with a prudent placement of a decimal point, we continue dividing until we either “get to the end”, or until the remainders begin to repeat.

For example, these are the decimal expansions of 1/8, 1/11, and 1/13:

$$
\begin{array}{r}
0.125 \\
8\,\overline{)1.000} \\
\underline{8} \\
20 \\
\underline{16} \\
40 \\
\underline{40} \\
0
\end{array}
\qquad
\begin{array}{r}
0.\overline{09} \\
11\,\overline{)1.00} \\
\underline{99} \\
1
\end{array}
\qquad
\begin{array}{r}
0.\overline{027} \\
37\,\overline{)1.000} \\
\underline{74} \\
260 \\
\underline{259} \\
1
\end{array}
\;.
$$

In the first case, we stopped because we arrived at a zero remainder. We say the the decimal expansion of 1/8 *terminates.* In the next two cases, we stopped because we arrived at a number which had been a previous remainder – this implies that the sequence of digits will repeat. This repetition goes on forever, and is indicated with the bar over the so-called “repeating part”.

To examine this more closely, lets look at trying to find the decimal expansion of 1/7.

$$
\begin{array}{r}
\\
7\,\overline{)1}
\end{array}
\quad
\begin{array}{r}
0.1 \\
7\,\overline{)1.0} \\
\underline{7} \\
3
\end{array}
\quad
\begin{array}{r}
0.14 \\
7\,\overline{)1.00} \\
\underline{7} \\
30 \\
\underline{28} \\
2
\end{array}
\quad
\begin{array}{r}
0.142 \\
7\,\overline{)1.000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
6
\end{array}
\quad
\begin{array}{r}
0.1428 \\
7\,\overline{)1.0000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
4
\end{array}
\quad
\begin{array}{r}
0.14285 \\
7\,\overline{)1.00000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
5
\end{array}
\quad
\begin{array}{r}
0.\overline{142857} \\
7\,\overline{)1.000000} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
1
\end{array}
$$

Look at the remainders we achieved: 1, 3, 2, 6, 4, 5, 1. We stopped because we looped back to the beginning. The question is this: during this process, did we *know absolutely for certain* that we would eventually either get to zero, or loop back to a previous remainder? The answer is yes: each of the remainders is a nonnegative integer which is less than the divisor (in this case the divisor is seven). So there are only six possible nonzero remainders. So either the expansion terminates, or it repeats. Moreover, the length of the repeating part is less than the divisor.

3. Exercises

Proposition 2. *Each reduced fraction has a decimal expansion which either terminates or repeats; moreover, the length of the repeating part is less than the denominator.*

Problem 1. Find the quotient and remainder when 2185 is divided by 13.

Problem 2. Use long division to write $\frac{10}{27}$ as a repeating decimal expansion.